p-Biharmonic Maps and Warped Product Manifolds

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Theorem 1. *Let (M,g) and (N,h) be two Riemannian manifolds. We denote by*

$$\begin{array}{rcccc} \mathbf{i}_{x_0} & : & (N,h) & \to & (M\times_{f^2} N, G_{f^2}), \\ & & y & \mapsto & (x_0,y) \end{array}$$

the inclusion map of N at the $x_0\in M$ level in $M\times_{f^2}N$. In the following, we characterize the p-biharmonicity of $\mathbf{i}_{x_0}$ in terms of the function f. The p-bitension field of the inclusion $\mathbf{i}_{x_0}$ is given by

$$\begin{aligned} \tau_{2,p}(i_{x_0}) &= -\frac{n^p}{8} f^{2(p-2)}(x_0)\left(\operatorname{grad}|\operatorname{grad} f^2|^2, 0\right)\circ \mathbf{i}_{x_0} \\ &\quad -\frac{(p-2)n^p}{8} f^{2(p-2)}(x_0)|\operatorname{grad} f^2|^2_{x_0}(\operatorname{grad} f^2, 0)\circ \mathbf{i}_{x_0}. \end{aligned}$$

The analogue of Theorem 1 was established in (see, [1])

Corollary 1. *The inclusion map $\mathbf{i}_{x_0}$ is p-biharmonic non p-harmonic if and only if x_0 is not a critical point for f^2 and at x_0*

$$\operatorname{grad}|\operatorname{grad} f^2|^2 + (p-2)|\operatorname{grad} f^2|^2 \operatorname{grad} f^2 = 0. \tag{1}$$

In the case where $|\operatorname{grad} f^2|^2$ is a function of f^2, we have the following result.

Corollary 2. *Let $f\in C^\infty(M)$ be a smooth positive function such that $|\operatorname{grad} f^2|^2 = e^{(2-p)f^2}$ on M. Then, every inclusion $\mathbf{i}_{x_0}$ is p-biharmonic non p-harmonic map.*

By using the above results, we obtain some new examples of p-biharmonic non p-harmonic maps for $p\geq 2$.

Example 1. Consider the inclusion map $\mathbf{i}_{x_0} : (N,h)\to(\mathbb{R}\times_{f^2} N, G_{f^2})$. Assume that $a=(f^2)'(x_0)>0$ and $b=(f^2)''(x_0)\leq 0$. By Corollary 1, $\mathbf{i}_{x_0}$ is p-biharmonic non p-harmonic if and only if $p = 2-\frac{2b}{a}$.

Example 2. Take M to be $(\frac{2}{p-2},\infty)\times\mathbb{R}^{m-1}$ equipped by the Riemannian metric $g=dt^2+dx_1^2+...+dx_{m-1}^2$. According to Corollary 2, every inclusion map $\mathbf{i}_{x_0}:(N,h)\to(M\times_{f^2}N, G_{f^2})$ is p-biharmonic non p-harmonic, where the function f^2 is given by

$$f^2(t,x) = \frac{1}{2-p}\ln\left(\frac{4}{(p-2)^2t^2}\right), \quad \forall (t,x)\in M.$$

Remark 1. Let (M,g) and (N,h) be two Riemannian manifolds. The inclusion map $\mathbf{j}_{y_0}$: $(M,g)\to(M\times_{f^2}N, G_{f^2})$ defined by $\mathbf{j}_{y_0}(x)=(x,y_0)$ is p-harmonic for all $y_0\in N$, because it is always a totally geodesic map, thus $\tau(\mathbf{j}_{y_0})=0$, and we have $|d\mathbf{j}_{y_0}|^2=m$, where m is the dimension of (M,g).

1. Balmuş A., Montaldo S., Oniciuc C. Biharmonic maps between warped product manifolds. J. Geom. Phys., 2007, 57, 449–466.